Rutgers University: Algebra Written Qualifying Exam

January 2006: Day 1 Problem 4 Solution

Exercise. Let A be a complex matrix of order n. Prove that A is nilpotent if and only if all of its eigenvalues are equal to zero.

Solution.

Let A be an $n \times n$ complex matrix.

A is nilpotent $\iff \exists k \in \mathbb{N}$ such that $A^k = 0$

($\Longrightarrow$) Suppose A is nilpotent.

Then $A^k = 0$ for some $k \in \mathbb{N}$

Let λ be an arbitrary eigenvalue of A

Then $\exists$a nonzero vector $\vec{v}$ s.t.

$$\begin{aligned} & A\vec{v} = \lambda\vec{v} \\ \Longrightarrow \quad & 0 = A^k\vec{v} \\ & \quad = A^{k-1}(A\vec{v}) \\ & \quad = \lambda A^{k-1}\vec{v} \\ & \quad \vdots \\ & \quad = \lambda^k\vec{v} \\ \Longrightarrow \quad & \lambda = 0 \qquad \text{since } \vec{v} \neq 0 \end{aligned}$$

Since λ was an arbitrary eigenvalue of A, all of the eigenvalues of A must be zero.

($\Longleftarrow$) If all of the eigenvalues of A are zero, then looking at the Jordan decomposition of A, we have

$$A = P \begin{bmatrix} 0 & a_1 & & \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1} \\ & & & 0 \end{bmatrix} P^{-1} \qquad \text{where } a_i = 1 \text{ or } 0$$

$$\Longrightarrow \quad A^n = P \begin{bmatrix} 0 & a_1 & & \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1} \\ & & & 0 \end{bmatrix}^n P^{-1}, \qquad \text{and} \qquad \underbrace{\begin{bmatrix} 0 & a_1 & & \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1} \\ & & & 0 \end{bmatrix}^n}_{\substack{\text{since it is an } n\times n \text{ upper triangular matrix} \\ \text{with 0 on its main diagonal}}} = 0$$

$\Longrightarrow \quad A^n = 0$

$\Longrightarrow \quad A$ is nilpotent